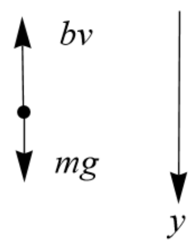
Problem 2.5

Suppose that a projectile which is subject to a linear resistive force is thrown vertically down with a speed v_{yo} which is *greater* than the terminal speed v_{ter}. Describe and explain how the velocity varies with time, and make a plot of v_y against t for the case that $v_{yo} = 2v_{\text{ter}}$.

Solution

Draw a free body diagram for the projectile. Let the positive y-direction be downward.



Apply Newton's second law in the y-direction to get the equation of motion.

$$\sum F_y = ma_y$$

The two forces to consider are the gravitational force and the linear air resistance force.
Let $v_y = v$.

$$mg - bv = m\frac{dv}{dt}$$

Add bv to both sides.

$$m\frac{dv}{dt} + bv = mg$$

Divide both sides by m.

$$\frac{dv}{dt} + \frac{b}{m}v = g$$

This is a first-order linear inhomogeneous ODE, so it can be solved with an integrating factor.

$$I = \exp\left(\int^t \frac{b}{m}\,ds\right) = e^{bt/m}$$

Multiply both sides by I.

$$e^{bt/m}\frac{dv}{dt} + \frac{b}{m}e^{bt/m}v = ge^{bt/m}$$

The left side can be rewritten as $d/dt(Iv)$ by the product rule.

$$\frac{d}{dt}(e^{bt/m}v) = ge^{bt/m}$$

Integrate both sides with respect to t.

$$e^{bt/m}v = \frac{mg}{b}e^{bt/m} + C$$

Solve for v by dividing both sides by $e^{bt/m}$.

$$v(t) = \frac{mg}{b} + Ce^{-bt/m}$$

In order to determine C, apply the initial condition $v(0) = v_{yo}$.

$$v(0) = \frac{mg}{b} + C = v_{yo}$$

Solve for C.

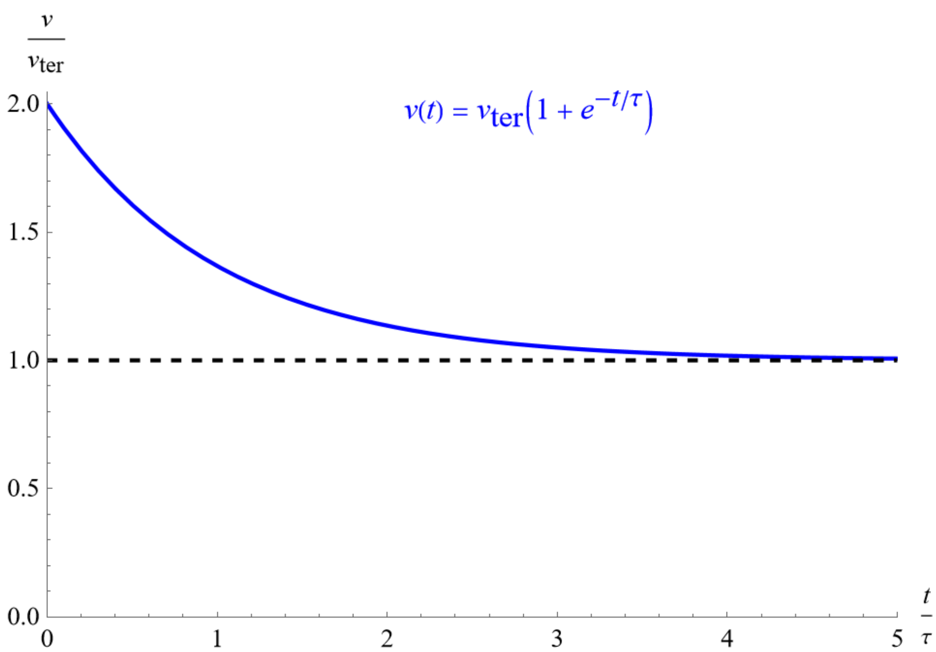
$$C = v_{yo} - \frac{mg}{b}$$

Therefore, the formula for the projectile's velocity is

$$v(t) = \frac{mg}{b} + \left(v_{yo} - \frac{mg}{b}\right) e^{-bt/m}.$$

If $v_{yo} = 2v_{\text{ter}} = 2mg/b$, then

$$\begin{aligned} v(t) &= \frac{mg}{b} + \left(\frac{2mg}{b} - \frac{mg}{b}\right) e^{-bt/m} \\ &= \frac{mg}{b} + \frac{mg}{b} e^{-bt/m} \\ &= \frac{mg}{b}(1 + e^{-bt/m}) \\ &= v_{\text{ter}}(1 + e^{-t/\tau}). \end{aligned}$$



The velocity of a projectile thrown downward with initial velocity greater than the terminal velocity will fall exponentially to the terminal velocity. This is because the air resistance is greater than the gravitational force.